Numerical Solution of the Shallow Water Equations

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Shallow Water Equations

Shallow water equations (SWEs) can be used to model fluid disturbances when the water depth is much less than wave length.

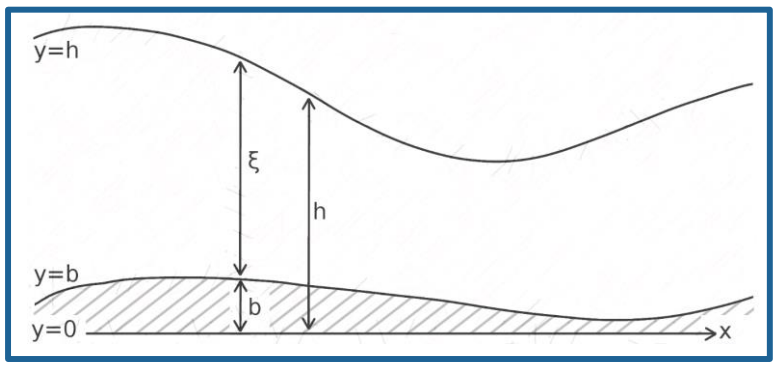


Figure 1

With the quantities as in Figure 1, the SWEs are:

$$\frac{\partial h}{\partial t} + \frac{\partial (u\xi)}{\partial x} = 0 \qquad (1)$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + g\frac{\partial h}{\partial x} = 0 \qquad (2)$$

These can be simplified when there is a flat bed, and applied to model a situation like a breaking dam. After the relevant simplifications, equations (1) and (2) can be combined into equation (3).

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0 \qquad (3)$$

$$U = \begin{bmatrix} h \\ q \end{bmatrix}, \; F(U) = \begin{bmatrix} q \\ \frac{q^2}{h} + \frac{1}{2}gh^2 \end{bmatrix} \qquad (4)$$

Dam Break

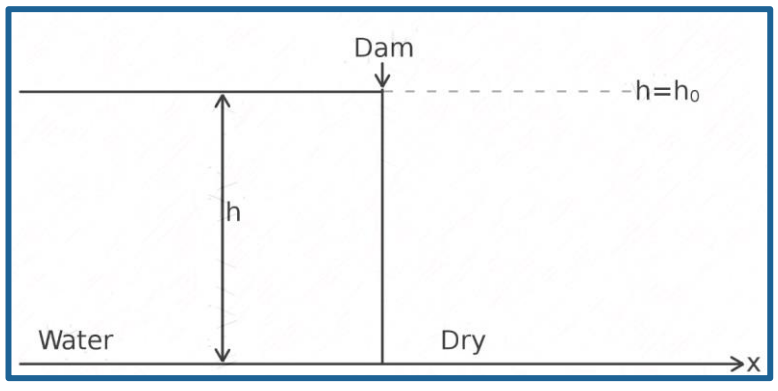


Figure 3

The dam break scenario sets a large expanse of still water to one side of a dam, and a shallow or dry region on the other. Then the dam is removed instantaneously. There is an analytical solution[1] to equation (3), but there are also many ways to solve it numerically, which are more versatile.

Two central schemes are the Lax-Friedrichs[2] and Kurganov-Tadmor[3] methods; equations (5) and (6) respectively. Clearly, the latter is more complex*. It is also much more accurate as the Lax-Friedrichs method has considerable dissipation of the wave.

$$U_i^{t+\Delta t} = \frac{1}{2}(U_{i+1}^t + U_{i-1}^t) - \frac{\Delta t}{2\Delta x}(F(U_{i+1}^t) - F(U_{i-1}^t)) \qquad (5)$$

$$U_i^{t+\Delta t} = \frac{1}{2}(U_{i+1}^t + U_{i-1}^t) - \frac{\Delta t}{\Delta x}(H_{i+1/2}^t - H_{i-1/2}^t) \qquad (6)$$

$$H_{i+1/2}(t) = \frac{F(U_{i+1/2}^+(t)) + F(U_{i+1/2}^-(t))}{2} - \frac{a_{i+1/2}(t)}{2}[U_{i+1/2}^+(t) - U_{i+1/2}^+(t)]$$

When implemented into a program, the accuracy of each can be analysed. Figure 2 shows the wave form estimated by the Lax-Friedrichs method compared to the analytical solution. Kurganov-Tadmor achieves even closer results.

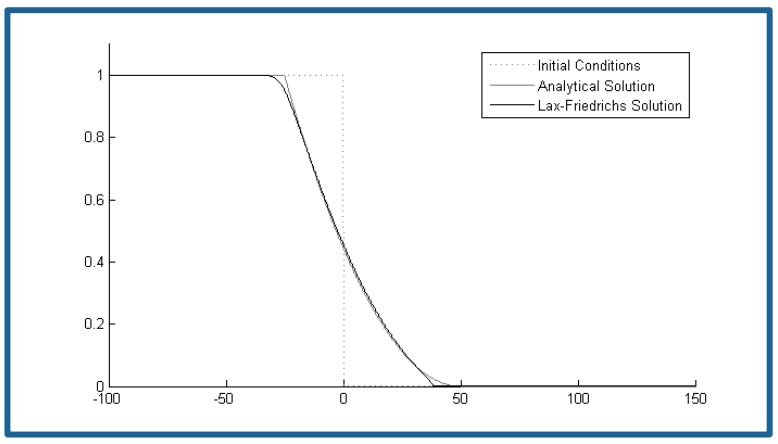


Figure 2

Tsunamis

SWEs are appropriate because tsunamis have such a long wave length by comparison to ocean. Reverting to the uneven bed equations (1) and (2), numerical solutions can be gained by using approximations for the effects the topography has on the surface wave. A propagating wave modelled like this may look like figure 4.

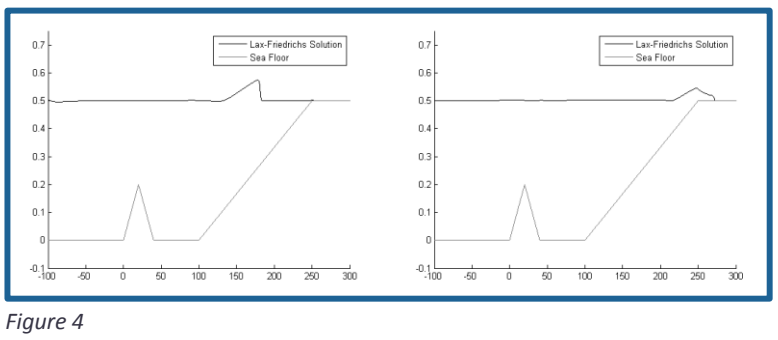


Figure 4

[1] A. Ritter *Die Fortpanzung der Wasserwellen* Verein Deutsch. Ing. Vol. 36, Issue 33, pp. 947-954 (1892) [2] R. J. LeVeque *Numerical Methods for Conservation Laws* Birkhuser (1992) p. 125 [3] Kurganov A. and Tadmor E. (2000) *New High-Resolution Central Schemes for Non-linear Conservation Laws and Convection-Diffusion Equations* Journal of Computational Physics, Vol. 160 Issue 1, pp. 241-282

* The ½ indexes are in reference to linearly interpolated cell boundary values, and $a_{i+1/2}$ is the local maximum wave speed in each cell division, for more see [3]